Soliton Collision Dynamics in the General Coupled Nonlinear Schrödinger System

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Abstract—We have obtained the bright one- and two-soliton solutions of the two-component general-coupled nonlinear Schrödinger equations by using the Hirota's bilinearization method. By studying the collision dynamics, we have pointed out that these bright solitons undergo two types of interesting shape changing collisions characterized by the energy redistribution and amplitude dependent phase-shifts which is not possible in their single component counterpart in addition to elastic collision and bound states.

Key words: Solitons, coupled nonlinear Schrödinger equation, Hirota's bilinearization method.

1. INTRODUCTION

Multicomponent soliton formation and their collision dynamics in nonlinear media is one of the main emphases of current research. The multifaceted applications of these multicomponent solitons make them as potential candidates for various physical processes in diverse areas of science [1, 2]. In the context of nonlinear optics, the multicomponent solitons are formed in optical fibers due to interplay between the dispersion/diffraction and nonlinear effects. In general cases like pico-second pulse propagation in non-ideal low birefringent multimode fibers or beam propagation in weakly anisotropic Kerr-type nonlinear media, the coherent nonlinear effects due to the interaction of co-propagating fields should also be considered in addition to the self-phase modulation and cross-phase modulation effects. Mathematically, optical soliton propagation in Kerr type nonlinear media can be well described within the framework of multicomponent nonlinear Schrödinger (NLS) equations. Although these equations are non-integrable in general they turn out to be integrable for specific choices of nonlinearity parameters [3–5].

This paper deals with the following integrable general-coupled nonlinear Schrödinger equations (g-CNLS) [5],

$$\begin{aligned} & iq_{1z}+q_{1tt}+2(a|q_1|^2+c|q_2|^2+bq_1q_2^*+b^*q_1^*q_2)q_1=0, \\ & iq_{2z}+q_{2tt}+2(a|q_1|^2+c|q_2|^2+bq_1q_2^*+b^*q_1^*q_2)q_2=0, \end{aligned} \tag{1}$$

arising in the context of pulse/beam propagation in nonlinear media. In eqn. (1) "*z*" and "*t*" are the propagation direction and retarded time respectively, q_1 and q_2 are slowly varying complex amplitudes in each polarization mode. Here *a*, *c* and *b* are the coefficients of self-phase modulation (SPM), cross-phase modulation (XPM) and four-wave mixing terms, respectively. The nonlinearity coefficients *a* and *c* are real while *b* is a complex parameter. The above system admits the following Lax pair and is shown to be integrable by inverse scattering transform (IST) method [5].

$$L=\begin{pmatrix} i\lambda & 0 & p \\ 0 & i\lambda & q \\ r_1 & r_2 & -i\lambda \end{pmatrix}, \qquad M=\begin{pmatrix} -2i\lambda^2-ipr_1 & -ipr_2 & -ip_x-2p\lambda \\ -iqr_1 & -2i\lambda^2-iqr_2 & iq_x-2q\lambda \\ -ir_{1x}-2\lambda r_1 & -ir_{2x}-2\lambda r_2 & ipr_1+iqr_2+2i\lambda^2 \end{pmatrix},$$

where $r_1=-ap^*-bq^*,\ r_2=-b^*p^*-cq^*$ and λ is a spectral parameter. In the absence of coherent coupling terms (i.e., four-wave mixing terms) the above system has been studied in [6–12] and different kinds of novel shape changing collisions of solitons characterized by the intensity redistribution, amplitude dependent phase shift and change in relative separation distance has been identified. This collision behaviour can be profitably used in soliton collision based optical computing and in soliton amplification.

Now it is of natural interest to investigate whether the shape changing collision still persists in the g-CNLS system (1) and if so how it is influenced by the coherent coupling term. To answer these questions, in this work, we have obtained the bright one- and two-soliton solutions by using the Hirota's bilinearization method and different types of soliton collision dynamics have been explored with the aid of the obtained solutions.

This paper is arranged in the following manner. In section 2 we obtain the bilinear equations of system (1) and the bright one- and two-soliton solutions are obtained in section 3. In section 4, three types of two-soliton collisions are discussed in addition to soliton bound states. Final section is devoted for conclusions.

2. HIROTA'S BILINEARIZATION METHOD

Hirota's bilinearization method [13] is one of the powerful tools to obtain the soliton solutions of integrable nonlinear systems. In this section, we obtain the bilinear equations of system (1) by this method. By performing the rational transformation,

$$q_1 = \frac{g}{f}, \qquad q_1 = \frac{h}{f}, \tag{2}$$

to eqn. (1), we obtain the following set of bilinear equations.

$$\begin{aligned} &D_1(g.f) = 0, \qquad D_1(h.f) = 0, \\ &D_2(f.f) = 2(a\,|\,g\,|^2 + c\,|\,h\,|^2 + bgh^* + b^* g^* h), \end{aligned} \tag{3}$$

where $D_1 = iD_z + D_t^2$, $D_2 = D_t^2$, g and h are complex functions while f is a real function, and * denotes the complex conjugate. The Hirota's bilinear operators D_z and D_t are defined as [13]

$$D_z^p D_t^z (a.b) = \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^p \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^q a(z,t) b(z',t')\Big|_{z=z',t=t'} \tag{4}$$

In order to solve the above set of bilinear equations (3), let us consider the following power series expansions for g, h and f:

$$g = \chi g_1 + \chi^3 g_3 + \ldots, \quad h = \chi h_1 + \chi^3 h_3 + \ldots, \quad f = 1 + \chi^2 f_2 + \chi^4 f_4 + \ldots, \tag{5}$$

where χ is the formal expansion parameter. The resulting set of equations, after collecting the terms with the same power in χ, can be solved recursively to obtain the exact forms of g, h and f.

3. BRIGHT SOLITON SOLUTIONS

In this section, we obtain the bright one– and two-soliton solutions by solving the above bilinear equations (3) recursively with help of power series expansion.

3.1 Bright One-Soliton Solution

To obtain the bright one-soliton solution of system (1) we restrict the power series expansion (5) as $g = \chi g_1$, $h = \chi h_1$, $f = 1 + \chi^2 f_2$. By substituting this expansion into the bilinear equations (3) and recursively solving the resulting set of linear differential equations, we get the one-soliton solution as

$$q_1 = \frac{\alpha_1 e^{\eta_1}}{1+e^{\eta_1+\eta_1^*+R_1}} \equiv A_1 k_{1R} \operatorname{sech}\left(\eta_{1R}+\tfrac{R_1}{2}\right) e^{i\eta_{1I}},$$
$$q_2 = \frac{\beta_1 e^{\eta_1}}{1+e^{\eta_1+\eta_1^*+R_1}} \equiv A_2 k_{1R} \operatorname{sech}\left(\eta_{1R}+\tfrac{R_1}{2}\right) e^{i\eta_{1I}}, \tag{6}$$

where $e^{R_1} = \frac{1}{k_1+k_1^*}(a|\alpha_1|^2 + c|\beta_1|^2 + b\alpha_1\beta_1^* + b^*\alpha_1^*\beta_1)$,

$$A_1 = \tfrac{\alpha_1}{2} e^{-\frac{R_1}{2}},\ A_2 = \tfrac{\beta_1}{2} e^{-\frac{R_1}{2}},$$
$$\eta_1 = k_1(t+ik_1 z), \eta_{1R} = k_{1R}(t+2k_{1I}z),\ \eta_{1I} = k_{1I}t + (k_{1R}^2 - k_{1I}^2)z,$$

The above one-soliton solution of system (1) is characterized by three arbitrary complex parameters α_1, β_1 and k_1. The amplitude of the soliton is q_j-th component is A_j, where $j = 1,2$. The speed and central position of the soliton is $2k_{1I}$ and $R_1/2k_{1R}$, respectively. Such a bright one-soliton of system (1) for the parameter choice $k_1 = 1-i, \alpha_1 = 1+i$ and $\beta_1 = 0.5-i$ is shown in Fig. 1.

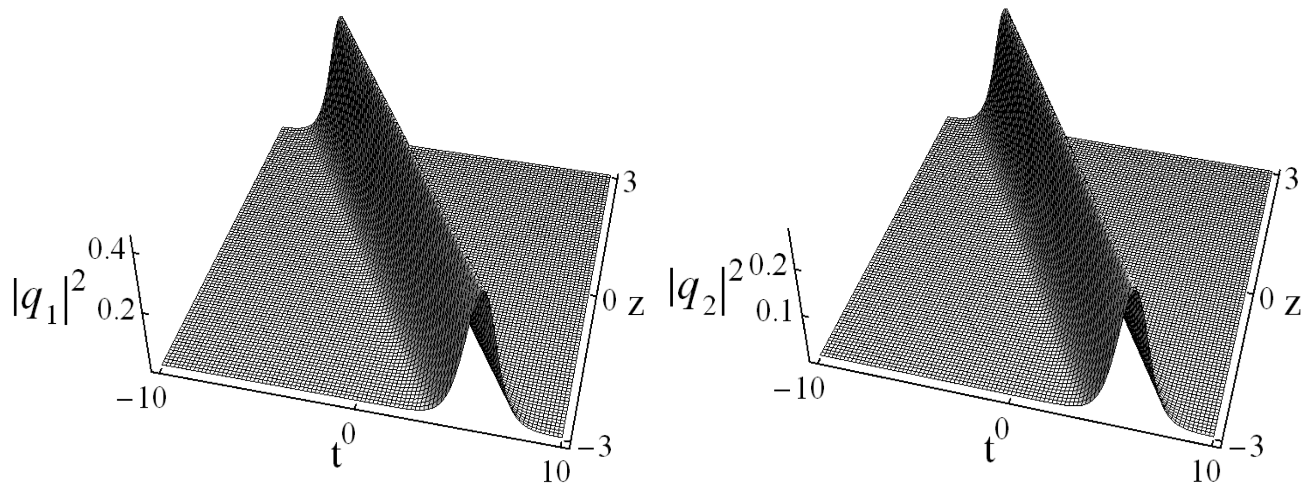

Fig. 1: Bright one-soliton solution of g-CNLS system (1)

3.2 Bright Two-Soliton Solution

The bright two-soliton solution of system (1) can be obtained by terminating the power series expansion (5) as, $g = \chi g_1 + \chi^3 g_3$, $h = \chi h_1 + \chi^3 h_3$, $f = 1 + \chi^2 f_2 + \chi^4 f_4$, and substituting this expressions into eqn.(3). After solving the resulting set of linear differential equations, we obtain the two-soliton solution as

$$q_1 = \tfrac{1}{D}(\alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + e^{\eta_1+\eta_1^*+\eta_2+\delta_1} + e^{\eta_1+\eta_2+\eta_2^*+\delta_2}),$$
$$q_2 = \tfrac{1}{D}(\beta_1 e^{\eta_1} + \beta_2 e^{\eta_2} + e^{\eta_1+\eta_1^*+\eta_2+\rho_1} + e^{\eta_1+\eta_2+\eta_2^*+\rho_2}), \tag{7}$$

where

$$D = 1 + e^{\eta_1+\eta_1^*+R_1} + e^{\eta_1+\eta_2^*+\delta_0} + e^{\eta_2+\eta_1^*+\delta_0^*} + e^{\eta_2+\eta_2^*+R_2} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3}.$$

Various quantities appearing in the above equation (7) are as below:

$$e^{R_1} = \frac{\kappa_{11}}{k_1+k_1^*},\ e^{\delta_0} = \frac{\kappa_{12}}{k_1+k_2^*},\ e^{R_2} = \frac{\kappa_{22}}{k_2+k_2^*},$$

$$e^{\delta_1} = \frac{(k_1-k_2)(\alpha_1\kappa_{21}-\alpha_2\kappa_{11})}{\left(k_1+k_1^*\right)\left(k_2+k_1^*\right)},\ e^{\delta_2} = \frac{(k_1-k_2)(\alpha_1\kappa_{22}-\alpha_2\kappa_{12})}{\left(k_1+k_2^*\right)\left(k_2+k_2^*\right)},$$

$$e^{\rho_1} = \frac{(k_1-k_2)(\beta_1\kappa_{21}-\beta_2\kappa_{11})}{\left(k_1+k_1^*\right)\left(k_2+k_1^*\right)},\ e^{\rho_2} = \frac{(k_1-k_2)(\beta_1\kappa_{22}-\beta_2\kappa_{12})}{\left(k_1+k_2^*\right)\left(k_2+k_2^*\right)},$$

$$e^{R_3} = \frac{(k_1-k_2)(k_1^*-k_2^*)(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21})}{\left(k_1+k_1^*\right)\left(k_1+k_2^*\right)\left(k_2+k_1^*\right)\left(k_2+k_2^*\right)},$$

and
$$\kappa_{ij} = \frac{a\alpha_1\alpha_1^*+c\beta_1\beta_1^*+b\alpha_1\beta_1^*+b^*\alpha_1^*\beta_1}{\left(k_i+k_j^*\right)}, \qquad j=1,2.$$

Here $\eta_j = k_j(t+ik_jz),\ j=1,2$. The real and imaginary parts of η_j are given by $\eta_{jR} = k_{jR}(t+2k_{jI}z)$ and $\eta_{jI} = k_{jI}t+(k_{jR}^2-k_{jI}^2)z,\ j=1,2.$ The above two-soliton solution is characterized by six arbitrary complex parameters α_j, β_j and k_j, $j=1,2.$

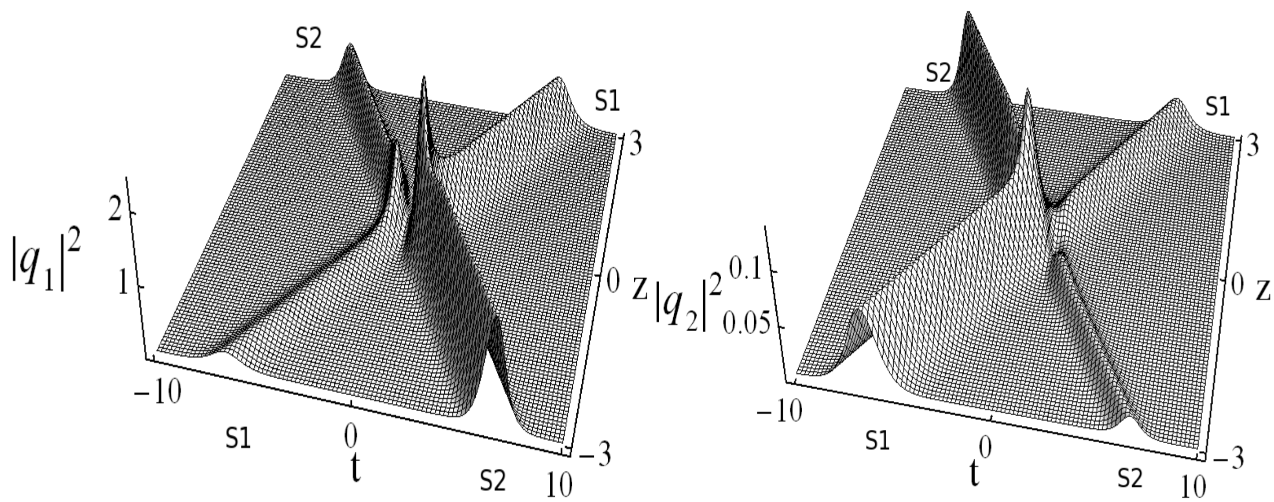

Fig. 2: Shape-changing collision of g-CNLS system (1) for a=c=*b=2.*

4. COLLISION OF SOLITONS

The two-soliton solution obtained in the previous section can be utilized to describe the collision of two bright solitons in the general-CNLS system (1). The detailed asymptotic analysis of two-soliton collision reveals that there exists two types of shape-changing (or) energy-sharing collisions corresponding to two different choices of SPM and XPM coefficients (i) *a,c>0* or *a,c<0* and (ii) *a>0 & c<0* or *a<0 & c>0* for arbitrary *b* values. In Fig. 2 we have shown the first type two-soliton collision for the parametric choices $a=c=b=2$, $k_1=1+i,\ k_2=1.5-i,\ \alpha_1=1+i,$ $\alpha_2=1-i,\ \beta_1=0.8+0.2i,$ and $\beta_2=0.5$. It can be observed that the collision scenario is intricate. During collision the soliton $S1$ undergoes an enhancement in its intensity while the soliton $S2$ gets suppressed in the q_1 component and the reverse scenario takes place in the q_2 component. This type of shape changing collision behaviour is not possible in the single component NLS system. Another type of shape-changing collision of two-soliton is given in Fig. 3 for *a=b=2* and *c=-2* with other parameters fixed as in Fig. 2. Here a given soliton shows same type of energy

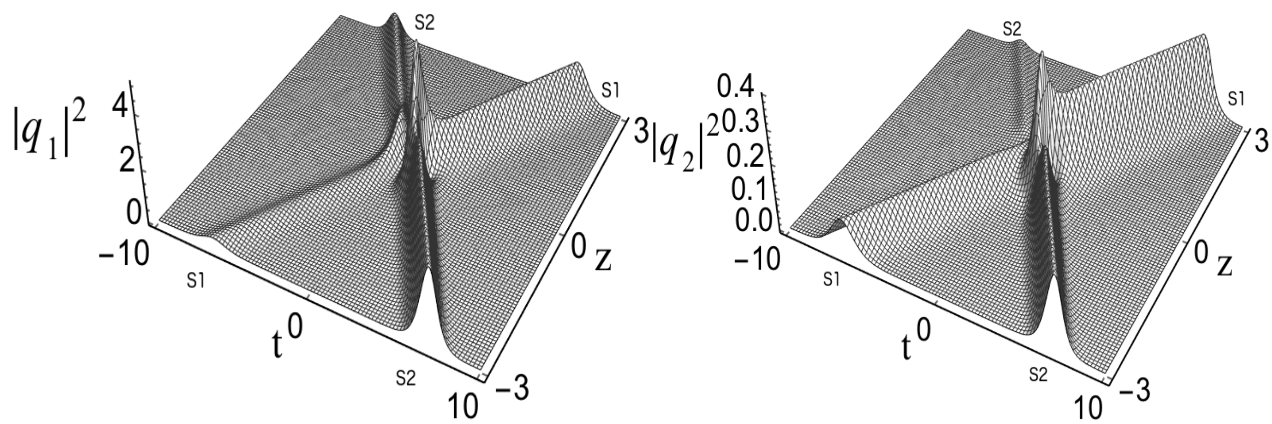

Fig. 3: Shape-changing collision of g-CNLS system (1) for a=2, c=-2, and *b*=*2.*

change in both components. Similar types of collision behaviour have also been observed in the Manakov system [8, 9] and in the mixed CNLS system [11, 12] in the absence of the phase dependent nonlinearity (*b=0*). The role of the phase dependent nonlinearities is to enhance the switching/sharing of intensities during collision. To facilitate the understanding, we present the soliton collision processes of the Manakov system in Fig. 4 for the same parameters but with *b=0*. The asymptotic analysis of the two soliton solution shows that the above shape-changing collisions are characterized by intensity redistribution among the solitons in both the components and amplitude dependent phase shifts resulting in the change in the relative separation distances between the solitons before and after collision. However the standard elastic collision with no alteration in the amplitudes of the colliding solitons, except for a phase-shift, occurs for the choice $\alpha_1 / \alpha_2 = \beta_1 / \beta_2$ and is shown in Fig. 5 for the following choice of parameters $a = c = b = 2$, $k_1 = 1 + i$, $k_2 = 1.5 - i, \alpha_1 = \alpha_2 = 1 + i, \beta_1 = \beta_2 = 2$. In addition to the above two-soliton collisions, one can obtain bound states among any number of solitons with same velocity/speed which can show either parallel propagation or breathing oscillations.

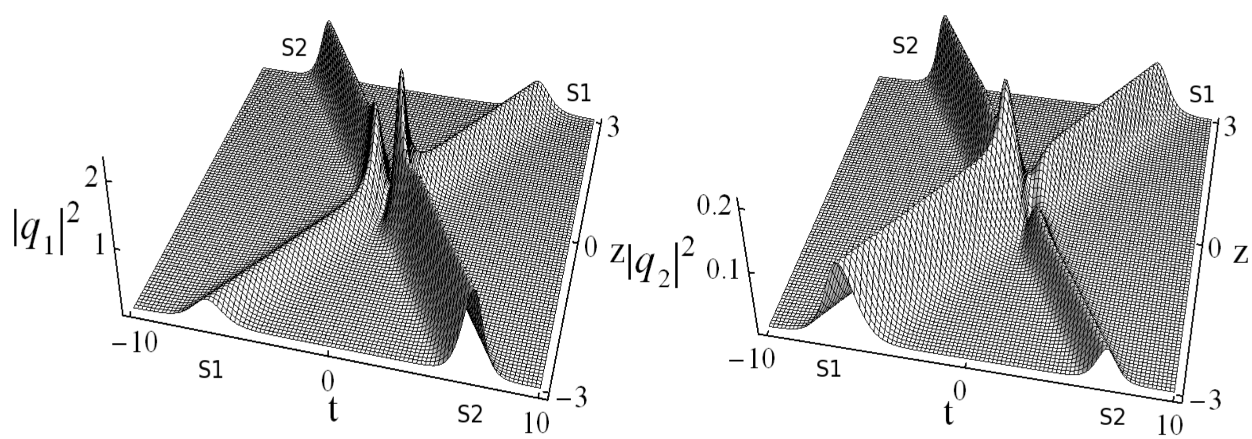

Fig. 4: Shape-changing collision of Manakov system (i.e. for *a=c=2* and *b=0* in system (1)).

5. Conclusion

The bright one- and two-soliton solutions of the two-component general-coupled nonlinear Schrödinger equations are obtained by using the Hirota's bilinearization method. We have identified two types of shape-changing collisions of bright solitons based on the SPM and XPM coefficients. Also, we found that still in the presence of coherent coupling terms the physically important shape-changing collision of multicomponent solitons persists and is enhanced further in a significant manner due to the effects of such phase dependent nonlinearities. This will find important ramifications in soliton collision based optical computing and in soliton amplification. The standard elastic collisions of solitons can also be observed for specific choice of the polarization parameters.

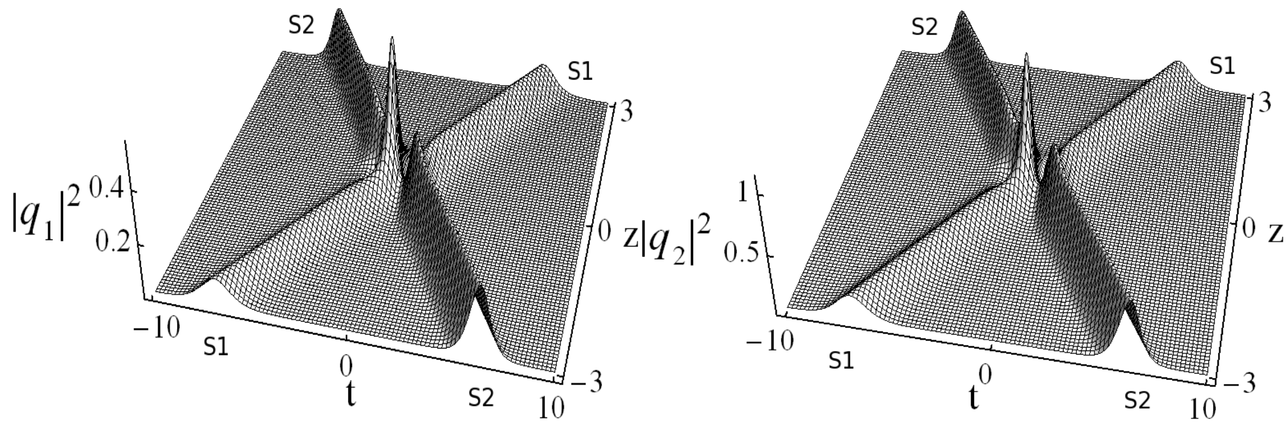

Fig. 5: Standard elastic collision of g-CNLS system (1).

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REFERENCES

[1] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, San Diego, 2003).
[2] N. Akhmediev and A. Ankiewicz, *Solitons: Nonlinear Pulses and Beams* (Chapman Hall, London, 1997).
[3] R. Radhakrishnan, R. Sahadevan, and M. Lakshmanan, Integrability and singularity structure of coupled nonlinear Schrödinger equations, *Chaos, Solitons Fractals* **5** (1995) 2315.
[4] Q-Han Park and H. J. Shin, Painlevé analysis of the coupled nonlinear Schrödinger equation for polarized optical waves in an isotropic medium, *Phys. Rev. E* **59** (1999) 2373.
[5] D. S. Wang, D. J. Zhang, and J. Yang, Integrable properties of the general coupled nonlinear Schrödinger equations, *J. Math. Phys.* **51** (2010) 023510.
[6] S. V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, *Zh. Eksp. Teor. Fiz.* **65** (1973) 505. [*Sov. Phys. JETP* **38** (1974) 248].
[7] R. Radhakrishnan, M. Lakshmanan, and J. Hietarinta, Inelastic collision and switching of coupled bright solitons in optical fibers, *Phys. Rev. E* **56** (1997) 2213.
[8] T. Kanna and M. Lakshmanan, Exact soliton solutions of coupled nonlinear Schrödinger equations: Shape-changing collisions, logic gates, and partially coherent solitons, *Phys. Rev. Lett.* **86**(2001)5043.
[9] T. Kanna and M. Lakshmanan, Exact soliton solutions of coupled nonlinear Schrödinger equations: Shape-changing collisions, logic gates, and partially coherent solitons, *Phys. Rev. E* **67** (2003)046617.
[10]T. Kanna, E. N. Tsoy, and N. Akhmediev, On the solution of multicomponent nonlinear Schrödinger equations, Phys. Lett. A **330** (2004) 224.
[11]T. Kanna, M. Lakshmanan, P. T. Dinda, and N. Akhmediev, Soliton collisions with shape change by intensity redistribution in mixed coupled nonlinear Schrödinger equations, *Phys.Rev.E* **73** (2006)26604.
[12]M. Vijayajayanthi, T. Kanna, and M. Lakshmanan, Bright-dark solitons and their collisions in mixed N-coupled nonlinear Schrödinger equations, *Phys. Rev. A* **77** (2008) 013820.
[13]R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, Cambridge, 2004).